An iterative method for computing the eigenproblem of Triangular Toeplitz matrices

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Contents

Max-plus algebra

Max-plus algebra - structure :

- $a \bigoplus b = \max(a, b)$
- $a \bigotimes b = a + b$
- $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$
- $\varepsilon = -\infty$, $e = 0$
- $(\mathbb{R}_{max}, \bigoplus, \bigotimes, \varepsilon, e)$

Eigenproblem

- Extending $\bigoplus$, $\bigotimes$ to matrices, vectors and polynomials over $\mathbb{R}_{max}$.
- At present the eigenproblem or maximum cycle mean problem is one of the most studied part of max-plus algebra.

$$A \otimes x = \lambda(A) \otimes x \quad (1)$$

Basic terms, examples

- By equation (1) we can describe the mathematical model of many practical problems. You can find practical examples and exact definitions of basic terms of max-plus algebra in book ⌈3⌋ B. Heidergott, G. J.Oldser, J. van der Woude: Max-plus at work. We can only illustrate some terms by the following example.

3	1	9	8
12	3	0	0
2	7	3	2
5	6	1	3

$L_\sigma = \{1,2,4\} \subseteq \{1,2,3,4\} = N, \sigma(1) = 4, ..$ - cycle

$w(\sigma, A) = 12 + 6 + 8$ - weight of cycle

A is an irreducible matrix iff

$(\forall i,j \in N)(\exists \sigma)(i,j \in L_\sigma; a_{k,\sigma(k)} \neq -\infty, k \in L_\sigma)$

Known results

Theorem ([1] Cuninghame-Green)

Any irreducible matrix **A** *has a unique eigenvalue*
$$\lambda(A) = \max_{\sigma \in P_{L_\sigma}} \frac{w(\sigma, A)}{|L_\sigma|},\ L_\sigma \subseteq \{1, 2, \ldots, n\}$$

- The Karp algorithm ([2] Karp) is a commonly accepted numerical method to solve the eigenproblem with complexity $O(n^3)$. It is based on calculation of the powers of matrix A.(in max-plus algebra)

Motivation

- Is there an algorithm, which is different and faster than the Karp algorithm or any other classical algorithms for computation of the eigenvalue ?
- In which classes of square matrices of n^{th} order can we find the eigenvalue faster than $O(n^3)$? (⌈5⌋ J. Plavka)

Triangular Toeplitz matrices

We considered the class of triangular Toeplitz matrices.

- The Toeplitz matrices are a special type of matrices, given by two vectors $u, v = (v_0, v_1, \ldots, v_{n-1})$.
- Vector u defines every above diagonal element and vector v every below diagonal element of the matrix A.
- If the vector u or v is the unit vector(not essential, that unit, but $v_0 < \lambda(A)$), except for the first element, then A is an *upper or lower triangular Toeplitz matrix*.
- A matrix A is a triangular Toeplitz matrix if it is an upper or a lower triangular.

Triangular Toeplitz matrices TTM

7	0	0	0		10	3	8	4
12	7	0	0		0	10	3	8
2	12	7	0		0	0	10	3
5	2	12	7		0	0	0	10

There is a *lower triangular Toeplitz matrix* (LTT) given by vector $v = (7, 12, 2, 5) = (v_0, v_1, v_2, v_3)$ $(a_{i,i-k} = v_k)$ on the left side of screen. The second matrix is an *upper triangular Toeplitz matrix* given by vector $u = (10, 3, 8, 4)^T$.

Lemma (1)

If $A = (a_{ij}) = A(v) \in \mathbb{LTT}^{n \times n}_{max}$ *then* $\lambda(A) = \lambda(A^T)$.

Basic properties of TTM

Theorem (2)

Let $A = (a_{ij}) = A(v) \in \mathbb{LTT}^{n\times n}_{max}$, $0 \leq z < \max_{k=1,\dots,n-1} v_k$

and the vector x(z) is given by relation :

a) $x_1(z) = 0$

b) $x_i(z) = \max\{x_{i-1}(z), \max_{j=1,\dots,i-1}\{v_{i-j} + x_j(z)\} - z\}$

for all $i = 2, \dots, n$*. Then* $\lambda(A) = z$ *and* $x(z)$ *is the unique fundamental eigenvector of matrix A if and only if* $z = x_n(z)$.

- The vector $x(z)$ with properties a) and b) from Theorem 2 is called **sub-eigenvector** in point z.

Basic properties of TTM

Any sub-eigenvector $x(z)$ can be associated to the **sub-critical permutation-**σ **.**

Definition (1)

σ is a sub-critical permutation of matrix A in point z if

1. $x_i(z) = a_{i,\sigma(i)} + x_{\sigma(i)}(z) - z$;
   holds for all $i \in L_\sigma$, such that $i > \sigma(i)$,
2. σ is a *cyclic permutation*,
3. $|L_\sigma| = min\{|L_\pi|;$ *permutation* $\pi : L_\pi \to L_\pi$ *with property 1) and 2)* $\}$

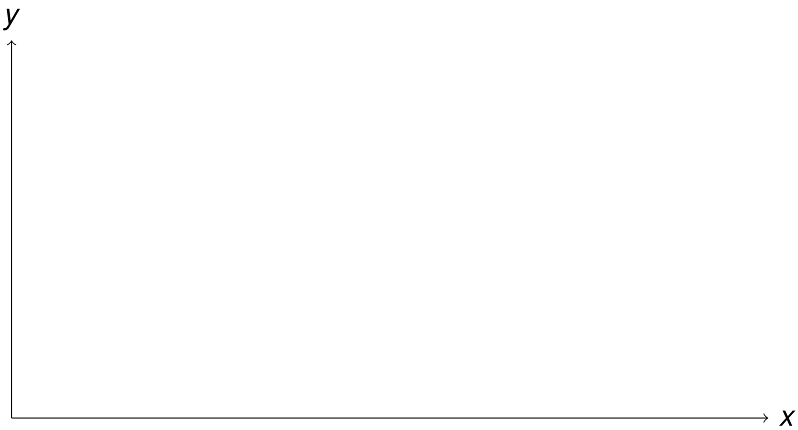
Basic properties of TTM

Between sub-eigenvector $x(z)$ and associated sub-critical permutation σ the relation

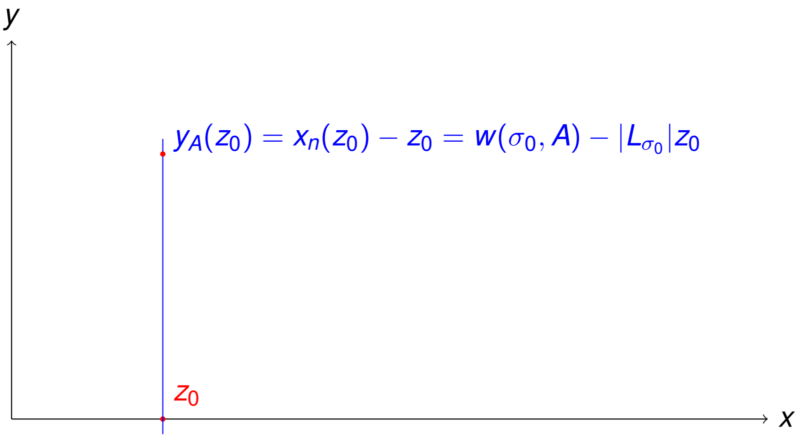
$$y_A(z_0) = x_n(z_0) - z_0 = w(\sigma, A) - |L_\sigma| z_0 \qquad (2)$$

is expressed, where $|L_\sigma|$ is the number of elements of permutation σ and $z_0 \in \langle 0, \max_{i=1,\dots,n-1} v_i)$. Permutation (cycle) σ defined as an **essential term** of function $y_A(z)$ in point z_0.([4] R.E. Burkard, P. Butkovič) In the next section we shaw essential terms of function $y_A(z)$.

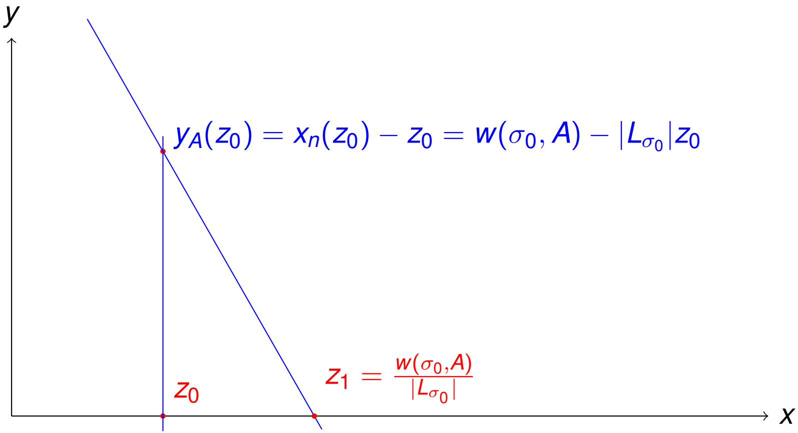
An iterative algorithm



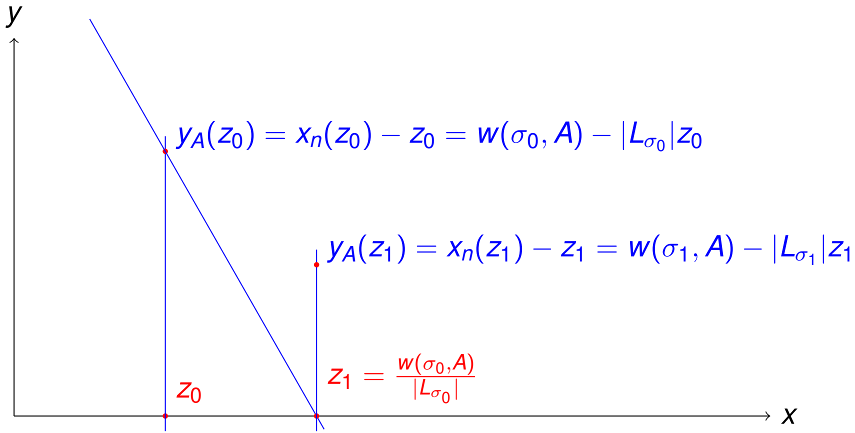
An iterative algorithm



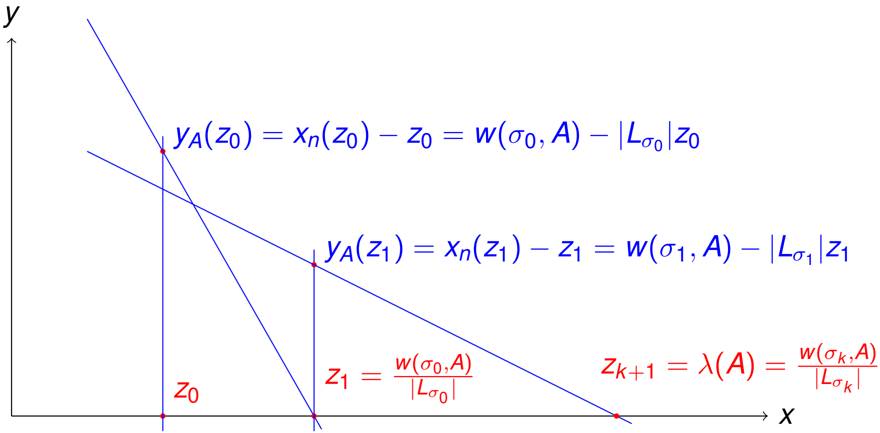
An iterative algorithm



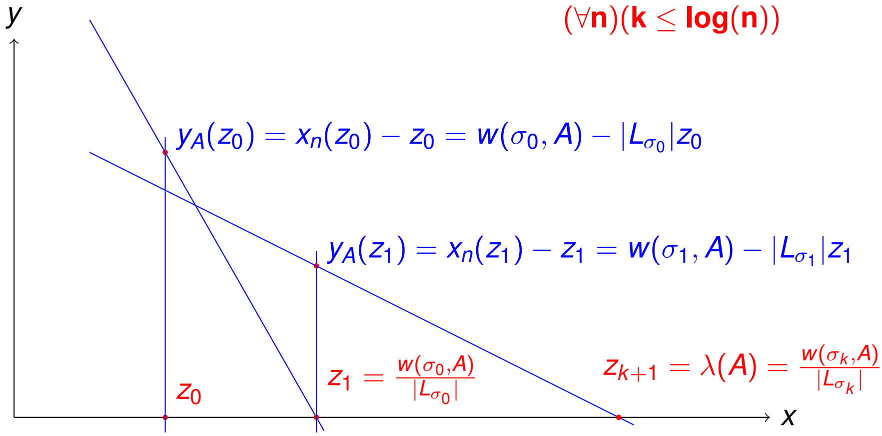
An iterative algorithm



An iterative algorithm



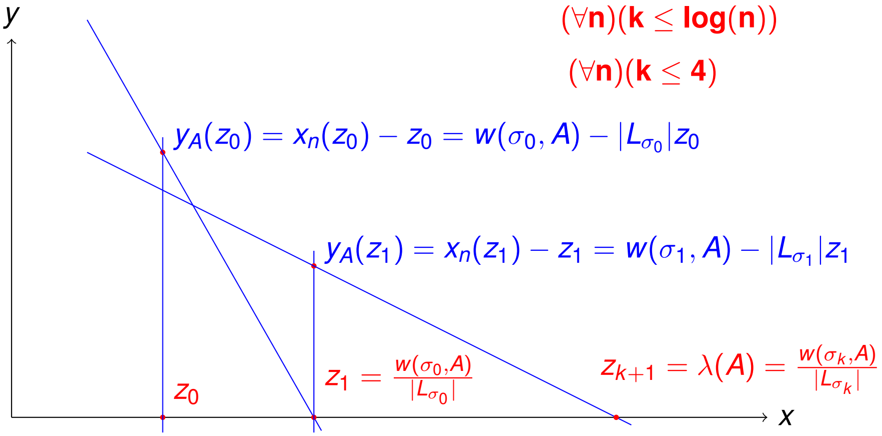
An iterative algorithm



An iterative algorithm

$$(\forall \mathbf{n})(\mathbf{k} \leq \mathbf{log}(\mathbf{n}))$$

$$(\forall \mathbf{n})(\mathbf{k} \leq \mathbf{4})$$



The pseudo-code of algorithm

Input: $A = (a_{ij}) = A(v) \in \mathbb{LTT}^{n\times n}_{max}$
(condition: $v_0 = a_{11} < \max\limits_{i=1,\dots,n-1} v_i$)

1. $z_0 = \max\limits_{i=1,\dots,n-1} (v_i/2)$; $k = 0$;
2. $z_k \mapsto x(z_k)$; // Theorem 2
3. $x(z_k) \mapsto \sigma_{k+1}$; // Definition of sub-critical permutation σ
4. $z_{k+1} = \frac{w(\sigma_{k+1}, A)}{|L_{\sigma_{k+1}}|}$;
5. if $z_{k+1} < x_n(z_{k+1})$ then $\{k = k+1; goto 2\}$;
6. $\lambda(A) = z_{k+1}$; $x(z_{k+1})$- the unique fundamental eigenvector.

Conclusion

- The complexity of iterative steps,to test whether the given number z_0 is the eigenvalue of the triangular Toeplitz matrix, equals to $O(n^2)$. (reckoning of sub-eigenvector and sub-critical permutation)
- We know to prove that the number of iterative steps is less or equal than $log(n)$, therefore the complexity of algorithm is $O(n^2 log(n))$.
- But, our calculations reveal that in the worst case scenario, the algorithm requires only four iterative steps to be performed.

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